JEE Main 2023 (Memory based)

1ˢᵗ February 2023 - Shift 2

Answer & Solutions

MATHEMATICS

1. If the term independent of x in the expansion of $\left(x^{\frac{2}{3}} + \frac{\alpha}{x^3}\right)^{22}$ is 7315, then $|\alpha|$ is:

   A. 1
   B. 2
   C. 0
   D. 3

   Answer (A)

   Solution:

   $$T_{r+1} = {}^{22}C_r \left(x^{\frac{2}{3}}\right)^{22-r} \left(\frac{\alpha}{x^3}\right)^r$$

   $$\Rightarrow \frac{2(22-r)}{3} - 3r = 0$$

   $$\Rightarrow 44 - 2r - 9r = 0$$

   $$\Rightarrow r = 4$$

   $$\therefore T_5 = {}^{22}C_4 \alpha^4 = 7315$$

   $$\Rightarrow \alpha^4 = \frac{7315}{7315} = 1$$

   $$\Rightarrow |\alpha| = 1$$

2. The value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} dx$ is:

   A. $\frac{3\pi^2}{\sqrt{6}}$

   B. $\sqrt{3}\pi^2$

   C. $\frac{\pi^2}{6\sqrt{3}}$

   D. $\frac{6\pi^2}{\sqrt{3}}$

   Answer (C)

   Solution:

   $$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x\,dx}{2 - \cos 2x} + \frac{\pi}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{2 - \cos 2x}$$

$= 0 + \frac{\pi}{4} \cdot 2 \int_0^{\frac{\pi}{4}} \frac{dx}{2 - \cos 2x}$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \frac{\sec^2 x \, dx}{1 + 3 \tan^2 x}$

Now, $\tan x = t$

$= \frac{\pi}{2} \int_0^1 \frac{dt}{1 + 3t^2}$

$= \frac{\pi}{2} \left[\frac{\tan^{-1}(\sqrt{3}t)}{\sqrt{3}} \right]_0^1$

$= \frac{\pi}{2\sqrt{3}} \left(\frac{\pi}{3} \right) = \frac{\pi^2}{6\sqrt{3}}$

**3.** The area determined by $xy < 8$, $y < x^2$ and $y > 1$ is:

A. $4 \ln 2 - \frac{14}{3}$

B. $4 \ln 2 + \frac{20}{3}$

C. $8 \ln 4 - \frac{14}{3}$

D. $8 \ln 4 - \frac{20}{3}$

Answer (C)

Solution:

Area $= \int_1^2 (x^2 - 1) \, dx + \int_2^8 \left(\frac{8}{x} - 1 \right) dx$

$= \frac{x^3}{3} - x \Big|_1^2 + (8 \ln x - x)|_2^8$

$= \left(\frac{8}{3} - 2 \right) - \left(\frac{1}{3} - 1 \right) + (8 \ln 8 - 8) - (8 \ln 2 - 2)$

$= \frac{4}{3} + 8 \ln 4 - 6$

$= 8 \ln 4 - \frac{14}{3}$
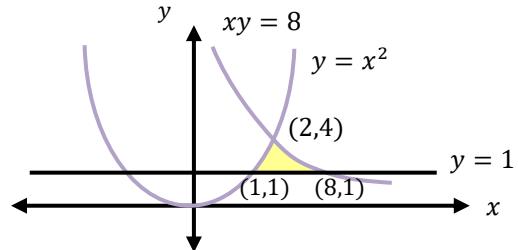
4. If $f(x) + f\left(\frac{1}{1-x} \right) = 1 - x$, then $f(2)$ equals:

A. $\frac{1}{4}$

B. $-\frac{5}{4}$

C. $\frac{3}{4}$

D. $-\frac{3}{4}$

Answer (B)

Solution:

$f(x) + f\left(\frac{1}{1-x} \right) = 1 - x \cdots (i)$

Put $x = 2$ in (i)

$f(2) + f(-1) = -1 \cdots (ii)$

Put $x = -1$ in (i)

$f(-1) + f\left(\frac{1}{2} \right) = 2 \cdots (iii)$

Put $x = \frac{1}{2}$ in (i)

$f\left(\frac{1}{2}\right) + f(2) = \frac{1}{2} \cdots (iv)$

$(ii) + (iv) - (iii)$ gives,

$f(2) + f(-1) + f\left(\frac{1}{2}\right) + f(2) - f(-1) - f\left(\frac{1}{2}\right) = -1 + \frac{1}{2} - 2$

$\Rightarrow 2f(2) = -1 + \frac{1}{2} - 2$

$\Rightarrow 2f(2) = -\frac{5}{2}$

$\Rightarrow f(2) = -\frac{5}{4}$

5. If $f(x) = x^x, x > 0$ then $f''(2) + f'(2)$ is:

   A. $10 + 12\ln 2 + 4(\ln 2)^2$

   B. $10 + 4(\ln 2)^2$

   C. $10 + 12\ln 2$

   D. $2^{\ln 2} + (\ln 2)^2$

Answer (A)

Solution:

$f(x) = x^x$

$f'(x) = x^x(1 + \ln x)$

$\therefore f'(2) = 4(1 + \ln 2)$

$f''(x) = \frac{x^x}{x} + x^x(1 + \ln x)^2$

$\Rightarrow f''(2) = 2 + 4(1 + \ln 2)^2$

$\Rightarrow f''(2) + f'(2) = 4 + 4\ln 2 + 6 + 8\ln 2 + 4(\ln 2)^2$

$\Rightarrow f''(2) + f'(2) = 10 + 12\ln 2 + 4(\ln 2)^2$

6. Which of the following is a tautology?

   A. $p \to (\sim p \land q)$
   B. $p \to (p \lor q)$
   C. $p \to (\sim p \lor q)$
   D. $p \to (\sim p \land \sim q)$

Answer (B)

Solution:

   a. $p \to (\sim p \land q)$

      $\cong (\sim p) \lor (\sim p \land q)$


   b. $p \to (p \lor q)$

      $\cong (\sim p) \lor (p \lor q)$
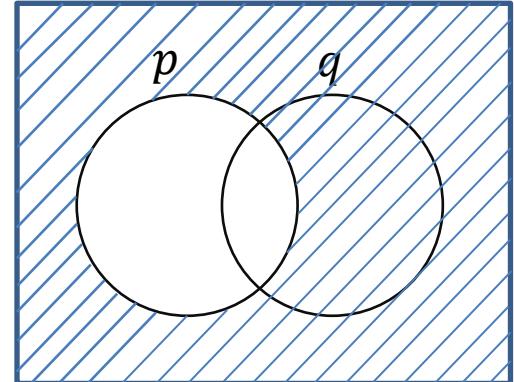
It can be inferred from the diagram it represents tautology.
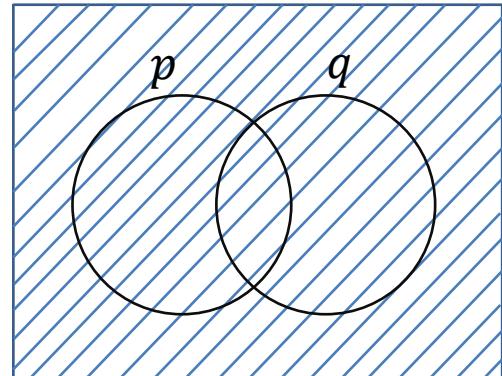
c. $p \to (\sim p \lor q)$

$\cong (\sim p) \lor (\sim p \lor q)$
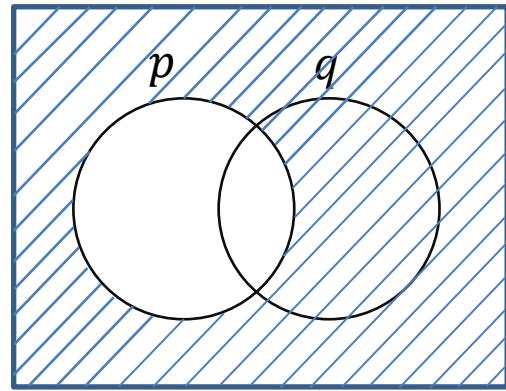
d. $p \to (\sim p \land \sim q)$

$\cong (\sim p) \lor (\sim p \land \sim q)$

$\cong (\sim p) \lor (p \lor q)'$
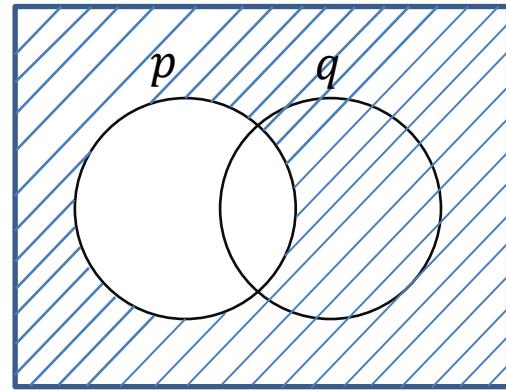
7. If the system of equations

$\alpha x + y + z = 1,$

$x + \alpha y + z = 1,$

$x + y + \alpha z = \beta$ has infinitely many solutions, then:

A. $\alpha = 1, \beta = 1$

B. $\alpha = 1, \beta = -1$

C. $\alpha = -1, \beta = -1$

D. $\alpha = -1, \beta = 1$

Answer (A)

Solution:

For infinite solutions

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

$\Rightarrow \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ \beta & 1 & \alpha \end{vmatrix} = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & 1 & 1 \\ 1 & \beta & \alpha \end{vmatrix} = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \beta \end{vmatrix} = 0$

Clearly $\alpha = 1 = \beta$ makes all the equations identical i.e., three coincidence planes.

$\therefore \alpha = 1 = \beta$

8. If $A = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$, then which of the following is true?

A. $A^{30} = A^{25}$

B. $A^{30} + A^{25} + A = I$

C. $A^{30} - A^{25} + A = I$

D. $A^{30} = A^{25} + A$

Answer (C)

Solution:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

(a) $A^{30} = (A^3)^{10} = (-I)^{10} = I$

$\qquad A^{25} = (A^3)^8 \cdot A = (-I) \cdot A = A$

$\qquad \Rightarrow A^{30} \neq A^{25}$

(b) $A^{30} + A^{25} + A = I + A + A = I + 2A \neq I$

(c) $A^{30} - A^{25} + A = I - A + A = I$

(d) $A^{30} - A^{25} - A = I - A - A = I - 2A \neq 0$

9. 2 unbiased die are thrown independently. A is the event such that the number on the first die is less than second die. B is the event, such that number on the first die is even and number on the second die is odd. C is the event such that first die shows odd number and second die shows even number. Then:

A. $n\big((A \cup B) \cap C\big) = 6$

B. A and B are mutually exclusive events

C. A and B are independent events

D. $n(A) = 18$, $n(B) = 6$, $n(C) = 6$

Answer (A)

Solution:

$A = \{(1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),(2,3),\ (2,4),\ (2,5),\ (2,6)(3,4),\ (3,5),\ (3,6)(4,5),\ (4,6),(5,6)\}$

$n(A) = 15$

$B = \{(2,1),\ (2,3),\ (2,5),\ (4,1),\ (4,3),\ (4,5),\ (6,1),\ (6,3),\ (6\ 5)\}$

$n(B) = 9$

$C = \{(1,2),(1,4),\ (1,6),\ (3,2),\ (3,4),\ (3,6),\ (5,2),\ (5,4),\ (5,6)\}$

$n(C) = 9$

$\big((A \cup B) \cap C\big) = \{(1,2),(1,4),(1,6),(3,4),(3,6),(5,6)\}$

$\Rightarrow n\big((A \cup B) \cap C\big) = 6$

$A \cap B = \{(2,3),\ (2,5),\ (4,5)\}$

$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$

$$P(A) = \frac{15}{36}, \; P(B) = \frac{9}{36}, \; P(C) = \frac{9}{36}$$

$$P(A) \cdot P(B) = \frac{15}{36} \cdot \frac{9}{36} = \frac{5}{48}$$

$\Rightarrow A$ and B are not independent events

10. If $\frac{dy}{dx} = \frac{x^2 + 3y^2}{3x^2 + y^2}$, $y(1) = 0$, then:

   A. $\frac{2x^2}{(x-y)^2} = \ln|x - y| + \frac{2x}{x-y}$

   B. $\frac{2x}{(x-y)^2} = \ln|x - y| + 1$

   C. $\frac{2x^2}{(x-y)^2} = \ln|x - y| + \frac{y}{x-y}$

   D. $\frac{2x}{(x-y)^2} = \ln|x - y| + \frac{y}{x-y}$

Answer (A)

Solution:

$\frac{dy}{dx} = \frac{x^2 + 3y^2}{3x^2 + y^2}$

Let $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{1 + 3v^2}{3 + v^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{1 + 3v^2}{3 + v^2} - v = \frac{-v^3 + 3v^2 - 3v + 1}{v^2 + 3}$

$\Rightarrow \frac{(v^2 + 3)}{-v^3 + 3v^2 - 3v + 1} dv = \frac{1}{x} dx$

$\Rightarrow \int \frac{(v^2 + 3)}{(1-v)^3} dv = \frac{1}{x} dx$

$\Rightarrow \int \frac{1}{(1-v)} dv - \int \frac{2}{(1-v)^2} dv + \int \frac{4}{(1-v)^3} dv = \int \frac{1}{x} dx$

$\Rightarrow -\ln|1 - v| - \frac{2}{(1-v)} + \frac{2}{(1-v)^2} = \ln|x| + C$

$\because y(1) = 0 \Rightarrow v(1) = 0 \Rightarrow C = 0$

$\therefore \frac{2}{\left(1 - \frac{y}{x}\right)^2} = \ln\left|1 - \frac{y}{x}\right| + \frac{2}{1 - \frac{y}{x}} + \ln x$

$\Rightarrow \frac{2x^2}{(x-y)^2} = \ln|x - y| + \frac{2x}{x-y}$

11. $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}, \vec{b} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}, \vec{c} = 4\hat{\imath} + 5\hat{\jmath} - \hat{k}$. If $\vec{r} \cdot \vec{b} = 0$ and $\vec{r} \times \vec{a} = \vec{b} \times \vec{c}$, then $\vec{r}$ is equal to:

   A. $-12\hat{\imath} - 8\hat{\jmath} + \hat{k}$
   B. $-12\hat{\imath} - \frac{23}{3}\hat{\jmath} + \hat{k}$
   C. $12\hat{\imath} + \frac{23}{3}\hat{\jmath} + \hat{k}$
   D. $12\hat{\imath} + 8\hat{\jmath} + \hat{k}$

Answer (B)

Solution:

$\vec{r} \times \vec{a} = \vec{b} \times \vec{c}$

$$\vec{b} \times (\vec{r} \times \vec{a}) = \vec{b} \times (\vec{b} \times \vec{c})$$
$$\Rightarrow (\vec{b}.\vec{a})\vec{r} - (\vec{b}.\vec{r})\vec{a} = (\vec{b}.\vec{c})\vec{b} - (\vec{b}.\vec{b})\vec{c}$$
$$\Rightarrow 6\vec{r} = -8(2\hat{\imath} - 3\hat{\jmath} + \hat{k}) - 14(4\hat{\imath} + 5\hat{\jmath} - \hat{k})$$
$$\Rightarrow 6\vec{r} = -72\hat{\imath} - 46\hat{\jmath} + 6\hat{k}$$
$$\Rightarrow \vec{r} = -12\hat{\imath} - \frac{23}{3}\hat{\jmath} + \hat{k}$$

12. If $2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \in (0, 1)$ has:

A. 2 solutions for $x < \frac{1}{2}$

B. 2 solutions for $x > \frac{1}{2}$

C. 1 solutions for $x < \frac{1}{2}$

D. 1 solutions for $x > \frac{1}{2}$

Answer (C)

Solution:

$$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $x = \tan\theta$ we get,

$$2\tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) = \cos^{-1}\cos 2\theta$$

$$\Rightarrow 2\left(\frac{\pi}{4} - \theta\right) = 2\theta$$

$$\Rightarrow \theta = \frac{\pi}{8}$$

$$\Rightarrow \tan^{-1} x = \frac{\pi}{8}$$

$$\Rightarrow x = \tan\frac{\pi}{8} = \sqrt{2} - 1$$

$$\therefore x = \sqrt{2} - 1 \text{ only 1 solution for } x < \frac{1}{2}$$

13. Number of non-negative integral solutions of $x + y + z = 21$ if $x \geq 1$, $y \geq 3$, $z \geq 6$ are _____.

Answer (78)

Solution:

$$\because x + y + z = 21 \quad [x \geq 1, y \geq 3, z \geq 6]$$
$$\Rightarrow (x - 1) + (y - 3) + (z - 6) = 11$$
$$\Rightarrow x_1 + y_1 + z_1 = 11$$
Where, $x_1 \geq 0$, $y_1 \geq 0$, $z_1 \geq 0$
Total $^{11+3-1}C_{3-1}$ solutions
$$^{13}C_2 = \frac{13!}{2!11!} = 6 \times 13 = 78$$

14. Total 6 digit numbers using the digits 4, 5, 9 which are divisible by 6 are _____.

Answer (81)

Solution:

We have,
For this, 4 will be fixed as unit place digit

		Total number
Case I:	$4's \to 6$ times	1

Case II: $4's \to 4$ times
$5's \to 1$ times $\frac{5!}{3!} = 20$
$9's \to 1$ times

Case III: $4's \to 3$ times
$5's \to 3$ times $\frac{5!}{2!3!} = 10$

Case IV: $4's \to 3$ times
$9's \to 3$ times $\frac{5!}{2!3!} = 10$

Case V: $4's \to 2$ times
$5's \to 2$ times $\frac{5!}{2!2!} = 30$
$9's \to 2$ times

Case VI: $4's \to 1$ times
$5's \to 1$ times $\frac{5!}{4!} = 5$
$9's \to 4$ times

Case VII: $4's \to 1$ times
$5's \to 4$ times $\frac{5!}{4!} = 5$
$9's \to 1$ times

Total numbers $= 81$

15. Let 3 $A.P.'s$ be

$S_1 = 2, 5, 8, 11, \cdots 394$

$S_2 = 1, 3, 5, 7, \cdots$

And $S_3 = 2, 7, 12, \cdots 397$
Then sum of common terms of these three $A.P.'s$ is _____.

Answer (2561)

Solution:

Common terms in S_1, S_2, S_3 are
$= 2, 17, 32, 47, \cdots$
S_2 has all odd numbers up to 397
Common terms in S_1, S_2, S_3 are
$= 17, 47, 77, \cdots, 377$
Sum of terms $= \frac{13}{2}(17 + 377)$
$= 2561$

16. Let $f(x) = |(x - 3)(x - 2)| - 3x + 2$ for $x \in [1,3]$. If M and m are absolute maximum & absolute minimum value of $f(x)$, then $|m| + |M|$ equals _____.

Answer (8)

Solution:

$|(x - 3)(x - 2)| = \begin{cases} (x - 2)(x - 3), & x \in [1,2) \\ -(x - 2)(x - 3), & x \in [2,3] \end{cases}$

$f(x) = \begin{cases} x^2 - 5x + 6 - 3x + 2, & x \in [1,2) \\ -x^2 + 5x - 6 - 3x + 2, & x \in [2,3] \end{cases}$

$$f(x) = \begin{cases} x^2 - 8x + 8, & x \in [1,2) \\ -x^2 + 2x - 4, & x \in [2,3] \end{cases}$$

$$f'(x) = \begin{cases} 2x - 8, & x \in [1,2) \\ -2x + 2, & x \in [2,3] \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} < 0, & x \in [1,2) \\ < 0, & x \in [2,3] \end{cases}$$

$\Rightarrow f'(x)$ is strictly decreasing in $[1,3]$

$f(1) = f(x)_{max} = M = 2 - 3 + 2 = 1$

$f(3) = f(x)_{min} = m = 0 - 9 + 2 = -7$

$\therefore |m| + |M| = |-7| + |1| = 8$

17. Let X_1, X_2, X_3, $\cdots$, X_7 is an A.P such that $X_1 < X_2 < X_3 \cdots < X_7$, $X_1 = 9$, $\sigma = 4$. The value of $\bar{X} + X_6$ is equal to _____.

Answer (34)

Solution:

Let the series be $a - 3d,\ a - 2d,\ a - d,\ a,\ a + d,\ a + 2d,\ a + 3d$

$a - 3d = 9$

Now if we shift the origin, the variance remains same

$\therefore$ for $-3d,\ -2d,\ -d,\ 0,\ d,\ 2d,\ 3d$

$\Rightarrow 16 = \frac{2}{7}(9d^2 + 4d^2 + d^2) - (\bar{X})^2$

$\Rightarrow 16 = \frac{2}{7}(14)d^2 - (0)^2$

$\Rightarrow d = 2$

$a - 3d = 9$

$\Rightarrow a = 15$

$\bar{X} = 15$

$X_6 = a + 2d = 19$

$\bar{X} + X_6 = 15 + 19 = 34$